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LOCALIZATION, COMPLETIONS AND METABELIAN GROUPS

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Abstract. For a pair of finitely generated residually nilpotent groups G, H, the group H is called para-G if there exists a homomorphism $G \to H$ which induces isomorphisms of all lower central quotients. Groups G and H are called para-equivalent if H is para-G and G is para-H. In this paper we consider the para-equivalence relation for the class of metabelian groups. For a metabelian group G, we show that all para-G groups naturally embed in a type of completion of the group G, a smaller and simpler analog of the pro-nilpotent completion of G, which is called the *Telescope of* G. This places strong restrictions on para-equivalent groups. In particular, for finitely generated metabelian groups, para-equivalence preserves the property of being finitely presented. Numerous examples illustrate our approach. We construct pairs of non-isomorphic para-equivalent polycyclic groups, as well as groups determined by their lower central quotients.

1. Introduction

In a recent paper [BMO1] entitled “A new look at finitely generated metabelian groups” we outlined a number of ideas for exploring finitely generated metabelian groups. These ideas arise from several seemingly different sources - algebraic geometry, algebraic number theory, combinatorial group theory and commutative algebra. Here we will concentrate on some of the details which involve localization and completions that were only briefly sketched in that paper. These completions take the form introduced in the important work of J. P. Levine [L1].

In 1935 W. Magnus proved that free groups are residually nilpotent. He used this theorem to deduce that an n-generator group, $n < \infty$, with the same lower central sequence as a free group of rank n, is free. D. Segal proved that a finitely generated metabelian group contains a subgroup of finite index which is residually nilpotent [Se]. The question arises as to whether certain residually nilpotent groups can be classified in terms of their lower central sequences. This suggests a possible approach to the Isomorphism Problem for finitely generated metabelian groups, one of the most tantalizing open problems in the study of finitely generated solvable groups. In large part, our work germinates from this insight. This program is outlined briefly in [BMO1].

1.1. **Definitions and notation.**

Let G be a group and let $x_1, x_2, \dots$ be elements of G. We denote the commutator $x_1^{-1}x_2^{-1}x_1x_2$ by $[x_1, x_2]$ and define, for $n > 0$,

$$[x_1, \cdots, x_{n+1}] = [[x_1, \cdots, x_n], x_{n+1}].$$

If H and K are subgroups of G, we define

$$[H, K] = gp([h, k] \mid h \in H, k \in K)$$

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Its subgroup $[G, G]$ is termed its derived group. G is *metabelian* if its derived group is abelian. The *lower central series*

$$G = \gamma_1(G) \geq \gamma_2(G) \cdots$$

of G is defined inductively by setting

$$\gamma_{n+1}(G) = [\gamma_n(G), G]$$

and the sequence

$$G/\gamma_2(G), G/\gamma_3(G), \ldots, G/\gamma_n(G), \ldots$$

is called *the lower central sequence of* G. A group G is *residually nilpotent* if

$$\bigcap_{n=1}^{\infty} \gamma_n(G) = 1.$$

1.2. **Residually nilpotent groups.** The lower central series appears naturally, not only in combinatorial group theory but also in many geometric problems which involve knots and links, arrangements of hyperplanes, homotopy theory, four manifolds and many other parts of mathematics. J. Stallings proved that the lower central sequence of a fundamental group is invariant under homology cobordism of manifolds. His result underpins Milnor and Massey product invariants of links, which have analogues for knots in arbitrary 3-manifolds using suitable variations of the lower central series. The homological properties of finitely generated parafree groups, i.e., those finitely generated residually nilpotent groups with the same lower central quotients as a free group, play an important role in low dimensional topology, see e.g., Cochran-Orr [CO].

A rather different class of residually nilpotent groups are the right-angled Artin groups many of which give rise to algorithmically unsolvable problems. In addition, many one-relator groups turn out to be residually nilpotent. The multitude of residually nilpotent groups constitute a large and important class of groups.

In particular every finitely generated metabelian group, the main concern of this paper, contains a residually nilpotent subgroup of finite index [Se].

1.3. **The pronilpotent completion.** Let G be a group and consider the set $\mathbb{N}$ of positive integers together with the unrestricted direct product

$$P = \prod_{n=1}^{\infty} G/\gamma_{n+1}(G).$$

Elements of P are functions

$$f : \mathbb{N} \longrightarrow \bigcup_{n=1}^{\infty} G/\gamma_{n+1}(G)$$

where $f(n) = g_n\gamma_{n+1}(G) \in G/\gamma_{n+1}, g_n \in G$. We use coordinate-wise multiplication, that is, the multiplication of such functions f, f' satisfies

$$(f.f')(n) = f(n)f'(n) = g_n\gamma_{n+1}(G)g'_n\gamma_{n+1}(G) = g_ng'_n\gamma_{n+1}(G).$$

Then the pronilpotent completion $\widehat{G}$ is, adopting the notation above, the subgroup of P defined by

$$\widehat{G} = \{f \in P \mid g_{n+1}\gamma_n(G) = g(n)\gamma_n(G) \text{ for all } n \in \mathbb{N}\}.$$

So for each n there exists $a_n \in \gamma_n(G)$ such that $g_{n+1} = g_n a_n$ and f can be thought of as an "infinite product"

$$f = g_1 a_1 a_2 \dots a_n \dots .$$

The subsequent well-known observation follows immediately from the definitions.

Lemma 1.1. *For any is residually nilpotent, G, the mapping*

$$\phi : g \mapsto (g\gamma_2(G), g\gamma_3(G), \dots, g\gamma_n(G), \dots)$$

embeds G into $\widehat{G}$.

An obvious question arises. Which of the properties of G does $\widehat{G}$ inherit? The following theorem of A. K. Bousfield [Bo1] fails, in general, without the finitely generated hypothesis.

Theorem 1.2 (A. K. Bousfield)**.** *If the residually nilpotent group G is finitely generated, then $\widehat{G}$ has the same lower central series as G.*

In the event that G is not finitely generated, the subgroup structure of $\widehat{G}$ can be extremely complicated even in the case where G is free. See, e.g., [BSta]. Some of the properties of a group do extend to its pronilpotent completion. For example if G is residually nilpotent and torsion-free, then $\widehat{G}$ is torsion-free.

We shall prove the following theorem in Section 3.

Theorem 1.3. *Let the metabelian polycyclic group G be residually nilpotent. Then the finitely generated subgroups of $\widehat{G}$ are again polycyclic.*

Theorem 1.3 depends on the following theorem, which is interesting in its own right.

Theorem 1.4. *A finitely generated metabelian group is polycyclic if and only if its two-generator subgroups are polycyclic.*

1.4. **Groups with the same lower central sequences.**

Definition 1.5. *Suppose that G and H are residually nilpotent groups. We term H* para-G *if there is a homomorphism of G into H which induces isomorphisms between the corresponding terms of their lower central sequences.*

G and H are termed para-equivalent *if H is para-G, and G is para-H.*

We emphasize that these are relations between *residually nilpotent* groups by definition. If H is para-G, then both G and H are residually nilpotent by hypothesis.

The proof of the following lemma is an easy consequence of the definition of a para-G-group H.

Lemma 1.6. *Suppose H is para-G. Then $\widehat{G} \cong \widehat{H}$.*

This is, of course, straight-forward. If H is para-G, then G and H have the same lower central series quotients, implying Lemma 1.6. But it has an interesting if also elementary consequence. Since $\widehat{G} \cong \widehat{H}$, and since G is residually nilpotent, G is isomorphic to a subgroup of $\widehat{H}$ by Lemma 1.1. Similarly, since H is residually nilpotent, Lemmas 1.1 and 1.6 imply that H is isomorphic to a subgroup of $\widehat{G}$. This is not sufficient to deduce that G and H are isomorphic, as the following theorem shows.

Theorem 1.7 (G. Baumslag, [Bau1]). *For each integer $n > 1$, there exist at least an infinite number of pair-wise non-isomorphic finitely generated residually nilpotent groups with the same lower central sequences as a free group of rank n.*

Residually nilpotent groups with the same lower central series as a free group, as noted earlier, are termed *parafree groups.* H. Neumann first speculated their existence more than 40 years ago, and asked whether Magnus' theorem could be extended to finitely generated residually nilpotent groups with the same lower central sequences as a free group. So H. Neumann's question was whether a finitely generated group with the same lower central sequence as a free group is free. Theorem 1.7 provides a negative answer to her question. Notice that if a finitely generated residually nilpotent group, G, has the same lower central sequence as a finitely generated free group F, there is a homomorphism of F into G which induces isomorphisms of the corresponding quotients $F/\gamma_n(F)$ and $G/\gamma_n(G)$. In other words G is para-F, i.e., parafree. More generally if G is free in some variety and G and H have the same lower central sequences, then there exists a homomorphism of G into H which induces an isomorphism on the corresponding terms of their lower central sequences, i.e., H is para-G.

1.5. **Para-equivalence of groups.** We analyze relations between the lower central sequence of a group and its isomorphism type by proving a type of *structure theorem* for finitely generated residually nilpotent, metabelian groups. We call this the Telescope Theorem. (See Theorem 2.1, as well as our prior paper [BMO1].) The Telescope Theorem embeds all para-G groups in a type of completion of the group G, a smaller and simpler analog of the pro-nilpotent completion of G which we call the *Telescope of* G.

In addition to Theorems 1.3 and 1.4 already stated, we prove the Telescope Theorem, and apply this to show the following:

i) We show that if G and H are para-equivalent, finitely generated metabelian groups, then G is finitely presented if and only if H is finitely presented. (See Theorem 2.14.)
ii) We show that if H is finitely generated and para-G, then G and H are para-equivalent. If G and H are finitely generated, para-equivalent metabelian groups, then G contains an isomorphic copy of H, and H of G. If G is paly-cyclic, these subgroups have finite index. (See Theorem 2.11 and Corollary 2.12.)
iii) Using ideal class theory, we give examples of non-isomorphic para-equivalent groups. These examples include polycyclic groups! (See Theorem 5.5 and Proposition 5.4.)
iv) We also give examples of groups determined by their lower central *sequences* in the class of finitely generated residually nilpotent groups. Let
$$\Gamma_n = \langle a, b \mid a^b = a^n \rangle,\ n \neq 2$$
and let H be a finitely generated residually nilpotent group with the same lower central sequence as Γ_n. Then $H \simeq \Gamma_n$ (Theorem 5.1).

Section 5 contains other interesting results and examples.

In general, this paper creates a foundation for further study of metabelian groups and there lower central series, and for the classification theorem, which will appear in [BMO2], of isomorphism classes of para-equivalent groups.

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2. Metabelian groups and telescopes

This section presents a proof of our *Telescope Theorem.* Some parts of this theorem below are due to J. P. Levine. (See Remark 2.2 below.)

For non-nilpotent residually nilpotent groups, $\hat{G}$ is always uncountable. Thus, the connection with G is less obvious, and more difficult to leverage. In [L1] and [L2], J. P. Levine defines closely related groups, his *algebraic closure of* G, whose image in the pronilpotent completion has important properties which prove invaluable in our work. In the special case where the group G is metabelian, we choose to call Levine's algebra closure of G the *Telescope of* G. This name highlights some fundamental properties of the algebraic closure of metabelian groups.

Our Telescope Theorem lays the foundation for our classification, using number theoretic tools, of para-equivalence classes of residually nilpotent, metabelian groups, which will appear in [BMO2].

Our fundamental and new contribution for metabelian groups is statement (i) from the Theorem below, which plays a crucial role in what follows.

Theorem 2.1. *(Telescope theorem) Let G be a metabelian, residually nilpotent group. Then there is a group G_S and a homomorphism $G \to G_S$ with the following properties:*

i) *There is an increasing sequence of subgroups*

$$G = G_1 \subset G_2 \subset G_3 \subset \cdots \subset G_S$$

with each $G_n \cong G$ and such that

$$G_S = \cup_n G_n$$

ii) *There is a short exact sequence:*

$$1 \to [G,G]_S \to G_S \to G_{ab} \to 1$$

where $[G,G]_S$ is the localization of $[G,G]$ as a $\mathbb{Z}[G_{ab}]$-module. (See Section 2.1.)

iii) *Each inclusion $G_n \to G_{n+1}$ induces an isomorphism on lower central series quotients:*

$$\frac{G_1}{\gamma_n G_1} \cong \frac{G_2}{\gamma_n G_2} \cong \frac{G_3}{\gamma_n G_3} \cong \cdots \cong \frac{G_S}{\gamma_n G_S}.$$

iv) *If H is finitely generated, then H is para-G via $\phi\colon G \to H$ if and only if ϕ induces an isomorphism $\overline{\phi}\colon G_S \to H_S$.*

v) *$G \mapsto G_S$ is functorial.*

We call the group G_S the *telescope of* G. (See Definition 2.8.)

Remark 2.2. The reader should compare this theorem with prior work of J. P. Levine. In [L1], Levine defined the closure of a group in its pronilpotent completion, which he also denoted G_S. (This is a quotient of the algebraic closure of groups which he later defined in [L2]. It has an additional condition on a normal generating set, but the minor modification we use here was familiar to Levine as well.) He computed this closure for *split* metabelian groups in Proposition 3.2 of [L3], and in this special split group case, Levine's computation coincides with ours. In particular, for split metabelian groups, parts ii) - v) above are not new.

Our primary new contribution is the addition of statement i).

2.1. **Properties of localization.** We start with recalling certain elementary properties of localizations which one can find in different textbooks, for example, in [AM].

Suppose R is a commutative ring with unit, M is an R-module, and S a multiplicative set containing the unit, $1 \in R$. We denote the result of inverting elements of S by M_S. Specifically, consider the abelian group

$$M_S := MS^{-1} = (M \times S)/\sim$$

where

$$(x, s_1) \sim (y, s_2)$$

if there is an element $s \in S$ such that

$$(xs_2 - ys_1)s = 0.$$

We denote the equivalence class of (x, s) by $\frac{x}{s}$, and the group law for M_S is given by

$$\frac{x}{s_1} + \frac{y}{s_2} = \frac{xs_2 + ys_1}{s_1 s_2}.$$

M_S is an R-module via the scalar action

$$\frac{x}{s} r = \frac{xr}{s}.$$

Note that R is a module over itself, and the R-module R_S has a natural ring structure given by

$$\frac{r}{s_1} \cdot \frac{t}{s_2} = \frac{rt}{s_1 s_2}.$$

Furthermore, M_S is an R_S-module via the action

$$\frac{x}{s_1} \cdot \frac{r}{s_2} = \frac{xr}{s_1 s_2}.$$

M_S is a flat R-module and a flat R_S-module. One easily checks, as well, that

$$M_S \cong M \underset{R}{\otimes} R_S$$

via the isomorphism

$$x \otimes \frac{r}{s} \mapsto \frac{xr}{s}.$$

We call an R-module M an S-local module if $M \cong M_S$ via the inclusion. Note that M is an S-local R-module if and only if the homomorphism $M \to M \otimes_R R_S$ defined by $x \mapsto x \otimes 1$ is an isomorphism. M_S is an S-local module.

For a group B, let $S = 1 + I$, where I is the augmentation ideal in $\mathbb{Z}[B]$. We observe some simple properties of $\mathbb{Z}[B]$-modules, for B abelian, and of localization:

(i) For any module M,

$$M_S \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z} \cong (M \otimes_{\mathbb{Z}[B]} \mathbb{Z}[B_S]) \otimes_{\mathbb{Z}[B]} \mathbb{Z} \cong M \underset{\mathbb{Z}[B]}{\otimes} ((\mathbb{Z}[B])_S \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z}) \cong M \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z},$$

since the image of any element of S under the augmentation homomorphism $\mathbb{Z}[B] \to \mathbb{Z}$ is $1 \in \mathbb{Z}$.

(ii) By exactness of localization, for any submodule $M \subset N$, we have $\frac{M_S}{N_S} \cong \left(\frac{M}{N}\right)_S$.

(iii) For any module M, $(MI)_S = (M_S)I$.

Lemma 2.3. *Let B be an abelian group, $f : A \to C$ be a homomorphism of $\mathbb{Z}[B]$-modules, such that the induced map $A/AI \to C/CI$ is an epimorphism and C is a finitely generated $\mathbb{Z}[B]$-module. Then the induced map $A_S \to C_S$ is an epimorphism.*

Proof. Let $c_1, \cdots c_k$ be generators for C, and define

$$\Phi\colon \oplus_{i=1}^k \mathbb{Z}[B] \to C$$

by $\Phi(e_i) = c_i$. Then the module homomorphism Φ is onto.

By hypothesis, there are elements a_i, $i = 1, \ldots, k$, with $f(a_i) - c_i = \Sigma_{j=1}^k c_j \beta_{ij} \in CI$. Define $\Psi\colon \oplus_{i=1}^k \mathbb{Z}[B] \to A$ by $\Psi(e_i) = a_i$, $i = 1, \cdots k$. This determines a commutative diagram of module homomorphisms as follows, where Λ is the square matrix over $\mathbb{Z}[B]$ with (i,j) entry given by $\delta_{ij} + \beta_{ij}$, and with Φ an epimorphism.

$$\begin{tikzcd}
\oplus_{i=1}^k \mathbb{Z}[B] \arrow[r, "\Lambda"] \arrow[d, "\Psi"] & \oplus_{i=1}^k \mathbb{Z}[B] \arrow[d, "\Phi"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

Note that the following is the identity isomorphism on $\mathbb{Z}^k$ since $\beta_{ij} \in \mathbb{Z}[B]I$, the augmentation ideal of $\mathbb{Z}[B]$.

$$\oplus_{i=1}^k \mathbb{Z}[B] \otimes_{\mathbb{Z}[B]} \mathbb{Z} \xrightarrow{\Lambda \otimes 1_{\mathbb{Z}}} \oplus_{i=1}^k \mathbb{Z}[B] \otimes_{\mathbb{Z}[B]} \mathbb{Z}.$$

Hence, the composition

$$\oplus_{i=1}^k \mathbb{Z}[B]_S \xrightarrow{S^{-1}\Lambda} \oplus_{i=1}^k \mathbb{Z}[B]_S \to S^{-1}C$$

is onto, as the first of these homomorphisms is an isomorphism. It follows that

$$S^{-1}A \xrightarrow{S^{-1}\Lambda} S^{-1}C$$

is onto as well. □

Lemma 2.4. *Let A be a $\mathbb{Z}[B]$-module where B is an abelian group, and suppose $S = 1 + \mathrm{Ker}\{\mathbb{Z}[B] \to \mathbb{Z}\}$. Fix $s, t \in S$. Then the homomorphism $a \mapsto a \cdot \frac{s}{t}$,*

$$A \to A_S$$

induces isomorphisms for all n

$$\frac{A}{AI^n} \to \frac{A_S}{(A_S)I^n}.$$

Proof. By properties (ii) and (iii) above,

$$\frac{A_S}{(A_S)I^n} \cong \frac{A_S}{(AI^n)_S} \cong \left(\frac{A}{AI^n}\right)_S.$$

By property (i),

$$\frac{A}{AI} \cong A \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z} \cong A_S \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z} \cong \frac{A_S}{(A_S)I}.$$

Here the middle isomorphism follows easily since $s, t \in S$ and these elements augment to 1.

Now assume the result holds for $n = k$. We have a commutative diagram where the second row is exact by the flatness of localization.

$$\begin{tikzcd}
0 \arrow[r] & \frac{AI^k}{AI^{k+1}} \arrow[r] \arrow[d] & \frac{A}{AI^{k+1}} \arrow[r] \arrow[d] & \frac{A}{AI^k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left(\frac{AI^k}{AI^{k+1}}\right)_S \arrow[r] & \left(\frac{A}{AI^{k+1}}\right)_S \arrow[r] & \left(\frac{A}{AI^k}\right)_S \arrow[r] & 0
\end{tikzcd}.$$

The right hand vertical homomorphism is an isomorphism by the inductive hypothesis. The following shows the left hand vertical homomorphism is also an isomorphism. In the equations below, once again the middle isomorphism follows since $s, t \in S$.

$$\frac{AI^k}{AI^{k+1}} \cong AI^k \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z} \cong (AI^k)_S \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z} \cong (A_S)I^k \underset{\mathbb{Z}[B]}{\otimes} \mathbb{Z} \cong \frac{(A_S)I^k}{(A_S)I^{k+1}} \cong \left(\frac{AI^k}{AI^{k+1}}\right)_S.$$

By the 5-Lemma, the result follows. $\square$

2.2. **The action on second cohomology.** Let B be an abelian group, and A a $\mathbb{Z}[B]$-module. Recall that two extensions $A \to G_i \to B$, $i = 1, 2$, are equivalent extensions if there is a diagram:

$$\begin{tikzcd}
 & & G_1 \arrow[dd, "\cong"] \arrow[dr] & & \\
1 \arrow[r] & A \arrow[ur] \arrow[dr] & & B \arrow[r] & 1 \\
 & & G_2 \arrow[ur] & &
\end{tikzcd}$$

We use the following well known result from the theory of group extensions. (See, for instance, [BSta].)

Theorem 2.5.

i) *There is a one-to-one correspondence $G \longleftrightarrow k(G)$*

$$\left\{\begin{matrix} \textit{Equivalence classes of} \\ \textit{extensions of the} \\ \mathbb{Z}[B]\textit{-module } A \textit{ by} \\ \textit{the abelian group } B \end{matrix}\right\} \longleftrightarrow H^2(B; A).$$

ii) *Given a homomorphism of $\mathbb{Z}[B]$-modules $\alpha\colon A \to M$, inducing a homomorphism of cohomology with twisted coefficients, the corresponding cohomology class*

$$\alpha(k(G)) \in Image\{H^2(B; A) \to H^2(B; M)\}$$

represents the extension G_α given as follows:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "\alpha_*"] & B \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & M \arrow[r, "j"] & G_\alpha \arrow[r] & B \arrow[r] & 1 \end{tikzcd}.$$

Here, $G_\alpha = (M \rtimes G)/U$ is a quotient of the semi-direct product by the normal subgroup $U \subset M \rtimes G$ given by $U = \{(\alpha(a), \iota(a^{-1})) \mid a \in A\}$.

Comment on notation: We clarify possible confusion arising above. If A is an abelian subgroup of a group G with abelian quotient B, then A is a $\mathbb{Z}[B]$-module where an element $b \in B$ acts on an element $a \in A$ by conjugating a by a lift of b to an element $\tilde{b} \in G$. That is, $a \cdot b := a^{\tilde{b}}$. For the group G_α given above, the submodule M is isomorphic to the abelian subgroup of G_α consisting of the elements $\{(m, 1) \mid m \in M\}$. The inclusion homomorphism $m \mapsto (m, 1)$ is a $\mathbb{Z}[B]$-module monomorphism since $(m, 1)^{\tilde{b}} = (m \cdot b, 1)$.

Lemma 2.6. *Let A be a $\mathbb{Z}[B]$-module, B an abelian group. Fix $g \in B$. Consider the $\mathbb{Z}[B]$-module homomorphism $\alpha\colon A \to A$ defined by $\alpha(a) = a \cdot g$ for all $a \in A$. This determines a coefficient induced homomorphism $\alpha\colon H^2(B; A) \to H^2(B; A)$ which is the identity homomorphism.*

Proof. Choose an element of $H^2(B; A)$ and represent this by an extension

$$1 \to A \to G \to B \to 1.$$

The homomorphism α induces a commutative diagram as follows, where $c_{\tilde{g}}$ is the inner automorphism given by conjugation by $\tilde{g}$.

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "c_{\tilde{g}}"] & B \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & A \arrow[r] & G_\alpha \arrow[r] & B \arrow[r] & 1 \end{tikzcd}$$

By construction this bottom extension in the above diagram is classified by $\alpha(k(G)) \in H^2(B; A)$.

We now define an equivalence of extensions between the extension G and G_α, thus proving that $\alpha\colon H^2(B; A) \to H^2(B; A)$ is the identity isomorphism as claimed.

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "="] & G \arrow[r] \arrow[d, "c_{\tilde{g}}"] & B \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & A \arrow[r, "\kappa"] & G_\alpha \arrow[r] & B \arrow[r] & 1 \end{tikzcd}$$

To do so, we must define the inclusion homomorphism κ making the above diagram commute. Define $\kappa(h) = (h, 1)^{\tilde{g}}$. Note that this is a $\mathbb{Z}[B]$-module homomorphism because B is an abelian group. In fact,

$$\kappa(a \cdot b) = (a \cdot b, 1)^{\tilde{g}} = ((a \cdot b) \cdot g, 1) = (a \cdot bg, 1) = (a \cdot gb, 1) = (a, g)^{\tilde{b}} = \kappa(a) \cdot b$$

where the fourth equality holds because B is abelian by hypothesis.

This concludes the proof.

□

Corollary 2.7. *Let $S = 1 + \ker\{\mathbb{Z}[B] \to \mathbb{Z}\}$ be the multiplicative set of elements of B which augment to 1. Then the action of S on $H^2(B; A)$ induced by multiplication by s, $s_*\colon A \to A$, satisfies $s_* = id\colon H^2(B; A) \to H^2(B; A)$.*

Proof. By the prior Lemma, the action of $g \in B$ on $H^2(B; A)$ is the identity isomorphism. Hence, for $g \in B$, $0 = (g-1)_*\colon H^2(B; A) \to H^2(B; A)$. It follows that for any $h \in I$, the augmentation ideal of $\mathbb{Z}[B]$, $h_* = 0$. Finally, this implies that for $s \in S$, $s_* = id$, as claimed. □

We now define the *Telescope of G*, for a residually nilpotent, metabelian group G.

Definition 2.8 (The Telescope of G)**.** *Let G be a residually nilpotent, metabelian group classified by an element $k(G) \in H^2(G_{ab}; [G, G])$. Define the* telescope of G, *G_S, to be the extension of $[G, G]_S$ by G_{ab} determined by the image of $k(G)$ under the coefficient induced homomorphism $H^2(G_{ab}; [G, G]) \to H^2(G_{ab}; [G, G]_S)$.*

Corollary 2.9. *Let $s \in S$. Then there is a isomorphism $\beta_s\colon G_S \to G_S$ such that:*

i) *The isomorphic image of G under β_s properly contains G.*
ii) *For any $g \in G_S$, there is an element $s \in S$ such that $g \in \beta_s(G)$.*

We denote the image, $\beta_s(G) \subset G_S$ by G_s.

Proof. Define a $\mathbb{Z}[G_{ab}]$-module homomorphism $\alpha_s\colon [G, G]_S \to [G, G]_S$ by $\alpha_s(a) = \frac{a}{s}$. Since s is invertible in $[G, G]_S$, this is an isomorphism. Furthermore, for $s \neq 1$, the image of α_s properly contains the module $[G, G]$.

To prove *i*), note that by Corollary 2.7, the isomorphism α_s determines an extension of $[G, G]_S$ by G_{ab},

$$1 \longrightarrow [G,G]_S \longrightarrow H \longrightarrow G_{ab} \longrightarrow 1 \ ,$$

and this is equivalent to the given extension G_S. (We avoid the use of double subscripts, and denote this group H instead of G_{α_s}.) Thus, we have two isomorphisms $G_S \cong H$. One is the homomorphism $\alpha_s\colon G_S \to H$. This gives us a commutative diagram as follows.

$$\begin{tikzcd} 1 \arrow[r] & {[G,G]_S} \arrow[r] \arrow[d, "\alpha_s"] & G_S \arrow[r] \arrow[d, "\alpha_s"] & G_{ab} \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & {[G,G]_S} \arrow[r] & H \arrow[r] & G_{ab} \arrow[r] & 1 \end{tikzcd} .$$

The other results from the equivalence of extensions, and makes the following commute.

$$\begin{tikzcd} 1 \arrow[r] & {[G,G]_S} \arrow[r] \arrow[d, "="] & G_S \arrow[r] \arrow[d, "\gamma"] & G_{ab} \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & {[G,G]_S} \arrow[r] & H \arrow[r] & G_{ab} \arrow[r] & 1 \end{tikzcd} .$$

So we have a composition

$$\beta_s = \gamma^{-1} \circ \alpha_s \colon G_S \to G_S.$$

Note that since $\beta_s(a) = \frac{a}{s}$ for every $a \in [G, G]$. So the image$(\beta_s([G, G]))$ properly contains $[G, G]$. As in the statement of the corollary, let G_s be the image of G under the isomorphism β_s. Then $G_s \cong G$. Furthermore, since $[G, G] \subsetneq \beta_s([G, G])$, $G \subsetneq G_s$, which completes the proof of statement i).

To see ii), suppose that $g \in G_S$. Recall that $G_S = ([G, G]_S \rtimes G)/\sim$ for an appropriate equivalence relation $\sim$. Suppose $g = [(a/s, h)]$, the equivalence class of an element $(a/s, h) \in [G, G]_S \rtimes G$. Then clearly $g \in \text{Image}(\beta_s)$ as claimed. □

Note 2.10. We note without proof that the construction of G_S does not depend on the metabelian decomposition of G, a result we will not use in this paper. That is, suppose that G is any metabelian group, and

$$1 \to A \to G \to B \to 1$$

is a short exact sequence with A and B abelian. Let H be the group classified by the image of $k(G)$ under the coefficient induced homomorphism $H^2(B; A) \to H^2(B; A_S)$. Then there is an isomorphism $H \to G_S$ making the following commute.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & H \arrow[dl] \\ G_S & \end{tikzcd}$$

2.3. **Proof of Telescope Theorem.**

Proof of the Theorem 2.1. Let G be a metabelian group G classified by an element

$$k(G) \in H^2(G_{ab}; [G, G]),$$

where an element $b \in G_{ab}$ acts on $[G, G]$ via conjugation by a lift of b into G. Recall that G_S is the extension of the module $[G, G]_S$ by G_{ab} given by the image of $k(G)$ under the coefficient induced homomorphism

$$H^2(G_{ab}; [G, G]) \to H^2(G_{ab}; [G, G]_S).$$

Here $S = 1 + \ker\{\mathbb{Z}[G] \to \mathbb{Z}\}$. Clearly G_S satisfies condition ii) of Theorem 2.1.

By Corollary 2.9, there are isomorphisms $\beta_s \colon G_S \to G_S$. Let G_s be the image of the restriction of β_s to the group $G \subset G_S$. By Corollary 2.9, these image subgroups G_s are isomorphic to G, contain G, and $G_S = \cup_s G_s$. Furthermore, enumerating the elenents of S, one readily verifies that if we set $G_n = G_{s_1 s_2 \ldots s_n}$ then the subgroups G_k form a cofinal sequence of subgroups whose union is G_S.

$$G \subset G_1 \subset G_2 \subset \cdots \cup_k G_k = G_S.$$

This proves statement i) of the Telescope Theorem.

One easily computes that $\gamma_k(G) = [G, G] \cdot I^{k-2}$. Hence *iii*) follows from Lemma 2.4.

Assume f induces isomorphisms on the lower central series quotients. Then $\gamma_k(G) = [G, G] \cdot I^{k-2}$, which implies that

$$\frac{[G, G]}{[G, G] \cdot I} \cong \frac{[H, H]}{[H, H] \cdot I}.$$

So Lemma 2.3 implies that

$$f_S\colon [G,G]_S \to [H,H]_S$$

is onto.

Since G and H are residually nilpotent, and since f induces an isomorphism on lower central series quotients, it follows that $G \to H$ is injective. Hence, $[G,G] \to [H,H]$ is injective. Since localization of commutative rings is flat it follows that

$$f_S\colon [G,G]_S \to [H,H]_S$$

is injective, and thus an isomorphism.

This implies $G_S \to H_S$ is an isomorphism. The other implication in *iv*) follows from *iii*).

Statement *v*) is immediate since localization of modules is functorial, and since G_S satisfies a type of pushout property. □

Theorem 2.11. *Let G be a metabelian, residually nilpotent group. If H is a finitely generated para-G-group, then G and H are para-equivalent. In particular, G is isomorphic to a subgroup of H, and H is isomorphic to a subgroup of G.*

Proof. By Theorem 2.1, iv), the homomorphism $G \to H$ we obtain from H being para-G induces an isomorphism of groups

$$G_S \simeq H_S.$$

Since the group H is finitely generated, the monomorphism $H \hookrightarrow H_S \simeq G_S = \cup_k G_k$ factors through some subgroup G_k, which by Theorem 2.1, part i), is isomorphic to G. Thus we have a commutative diagram of groups and homomorphisms as follows:

$$\begin{tikzcd} H \arrow[r, tail] \arrow[d] & G_S \\ G \arrow[ur, tail] & \end{tikzcd} .$$

We denote this vertical map by $h : H \to G$. Clearly, h is a monomorphism since h is a monomorphism. h induces isomorphisms of lower central quotients as claimed.

Furthermore, since the homomorphism $G \to H$ and the homomorphism h are both injective, the final claim of the Theorem holds as well. □

Corollary 2.12. *Let G, H and f be the same as in Theorem 2.11. Suppose that G is a polycyclic group. Then H is polycyclic, and isomorphic to a subgroup of G of finite index. Similarly G is isomorphic to a subgroup of H of finite index.*

Proof. Theorem 2.11 gives a sequence monomorphisms $[H,H] \hookrightarrow [G,G] \hookrightarrow [H,H]$. Since G is polycyclic, $[G,G]$ is a finitely generated abelian group, hence the above monomorphism $[H,H] \to [H,H]$ has a finite cokernel and the result follows. □

In spite of the strong and perhaps surprising Corollary, we will show in Section 5 that par-equivalent groups G and H need not be isomorphic, even when G and H are polycyclic.

Corollary 2.13. *Let G and H be finitely generated residually nilpotent groups. Then if G is free metabelian and H is para-G, then H is isomorphic to a subgroup of G.*

Corollary 2.13 is surprising in that it demonstrates the unexpected existence of a host of subgroups of even the two-generator free metabelian group. So, for instance, the non-abelian parafree metabelian group generated by a, b, c and satisfying the relation $a^2b^2 = c^n$ where n is odd, is a subgroup of a free metabelian group of rank two. A sketch of the proof of the existence of these parafree metabelian groups can be found in [Bau2]. The existence of such relations in ordinary free groups were first considered by R. Vaught and R. Lyndon in connection with the so-called Tarski problem. It turns out that such equations can only hold trivially in ordinary free groups. The Tarski problem was solved several years ago independently by O. Kharlampovich and A.Miasnikov [KM] on the one hand and Z. Sela [Sela] on the other.

2.4. **Finite presentation and completion.** We prove that if two groups are para-equivalent and finitely generated, they are either both *finitely presented* or neither is finitely presented.

Recall the formulation of a result of R. Bieri and R. Strebel [BStr]. Let B be an abelian group and $v \in Hom(B, \mathbb{R})$. Define the submonoid $B_v := \{b \in B \mid v(b) \geq 0\}$. Now let A be a $\mathbb{Z}[B]$-module. Then, for every $v \in Hom(B, \mathbb{R})$, A can be viewed as a module over the commutative ring $\mathbb{Z}[B_v]$. A finitely generated $\mathbb{Z}[B]$-module A is called *tame* if, for every $v \in Hom(B, \mathbb{R})$, either A is finitely generated as a $\mathbb{Z}[B_v]$-module, or else it is finitely generated as a $\mathbb{Z}[B_{-v}]$-module.

To prove Theorem 2.14 below, we apply the following theorem of R. Bieri and R. Strebel.

The Bieri-Strebel Theorem. [BStr] *Let*

$$1 \to A \to G \to B \to 1$$

be an exact sequence of groups, where A, B are abelian and G finitely generated. Then G is finitely presented if and only if A is a tame $\mathbb{Z}[B]$-module.

The main property of tame modules which we will use here is that every submodule of a tame module is tame.

Theorem 2.14. *Let G and H be finitely generated, residually nilpotent, metabelian groups. Suppose H is a para-G group. Then G is finitely presented if and only if H is finitely presented.*

Proof. Since H is finitely generated, G and H are para-equivalent by Theorem 2.11. In particular, G and H are isomorphic to subgroups of each other. In particular, there are monomorphisms of $\mathbb{Z}[G_{ab}] \simeq \mathbb{Z}[H_{ab}]$-modules

$$[G, G] \hookrightarrow [H, H] \hookrightarrow [G, G].$$

By Bieri-Strebel Theorem, G is finitely presented if and only if H is. □

Remark 2.15. Theorem 2.14 can not be generalized to the class of all finitely generated nilpotent groups. In a recent paper [BR], M. Bridson and A. Reid constructed a pair of finitely generated residually nilpotent groups G, H, such that H is para-G, H is finitely presented but G is not.

3. The proofs of Theorem 1.3 and 1.4

The proof of Theorem 1.3 depends on Theorem 1.4 which we prove first.

3.1. **The proof of Theorem 1.4.** Suppose that the two-generator subgroups of the finitely generated metabelian group G are polycyclic and that G is generated by $x_1, \dots, x_\ell$. Then the commutator subgroup $A = [G, G]$ of G is the normal closure of finitely many elements, say $a_1, \dots, a_m$. Now the subgroup of G generated by a_1 and x_1 is polycyclic. So in particular then the subgroup A_1 of A generated by the conjugates of a_1 by the powers of x_1 is also finitely generated, say by the elements $a(1,1), \dots, a(1,n_1)$. Now the subgroup of A generated by the conjugates of the finitely many elements $a(1,1), \dots, a(1,n_1)$ by the powers of x_2 is again finitely generated. Iterating this process we find that the subgroup B of A generated by the conjugates of the elements $a_1, \dots, a_m$ by the finitely many elements $x_1, \dots, x_\ell$ is also finitely generated. But $B = A$, the derived group of G. Consequently G is an extension of one finitely generated abelian group by another and is therefore polycyclic. So this completes the proof of Theorem 1.4.

3.2. **The proof of Theorem 1.3.** We will need the following lemma as preparation for the proof.

Lemma 3.1. *Let G be a polycyclic metabelian group. Let A be an abelian normal subgroup of G with abelian factor group $Q = G/A$. View A as a module over the integral group ring R of Q. Then for each $t = sA \in Q, a \in A$, there exist polynomials*

$$\alpha = c_0 + c_1 t + \cdots + c_{m-1} t^{m-1} - t^m$$

and

$$\beta = -t^{-1} + d_0 + d_1 t + \cdots + d_{n-1} t^{n-1}$$

such that

$$a\alpha = a\beta = 0.$$

Moreover given two such polynomials α and β if $a\alpha = a\beta = 0$, then it follows that the conjugates of a by the powers of s generate an abelian group which can be generated by $m + n$ elements.

Proof. (1) Consider the subgroup B_i of the subgroup $gp(a, s)$ of G generated by

$$a, a^s, \dots, a^{s^i}.$$

Then since $gp(a, s)$ is polycyclic, it satisfies the maximal condition, i.e. every subgroup is finitely generated. So there exists an integer m such that $a^{s^m} \in B_{m-1}$. Hence there exists $c_0, \dots, c_{m-1}$ such that

$$a^{s^m} = a^{c_0} + a^{c_1 s} + \cdots + a^{c_{m-1} s^{m-1}}.$$

Hence $a\alpha = 0$ as claimed.

(2) Consider the subgroup C_j generated by

$$a^{s^{-j}}, a^{s^{-j}+1}, \dots, a^{s^{-1}}.$$

Since G satisfies the maximal condition there exists an integer n such that

$$a^{s^{-n}} \in C_{n-1}.$$

So there exist $d_0, d_1, \dots d_n$ such that

$$a^{s^{-n}} = a^{d_0 s^{-n+1}} + a^{d_1 s^{-n+2}} + \cdots + a^{d_s^{-1}}.$$

Since the action of t on A is by conjugation by s, we can re-express what we have proved by writing $a\beta = 0$ as claimed

□

Now let G be a residually nilpotent, polycyclic metabelian group. Our objective is to prove that the finitely generated subgroups of $\widehat{G}$ are polycyclic. In view of Theorem 1.4 it suffices to prove that the two-generator subgroups of $\widehat{G}$ are polycyclic. The following simple lemma will facilitate the proof.

Lemma 3.2. *Let H be a metabelian group generated by the elements s and a. Then H is polycyclic if the subgroup generated by conjugates of $[s,a]$ by the powers of s is finitely generated and the subgroup generated by the conjugates of $[s,a]$ by the powers of a is finitely generated.*

Proof. Notice that $[H,H]$ is the normal closure in H of $[s,a]$. Let $h_1,\dots,h_m$ be a finite set of generators of the subgroup of H generated by the conjugates of $[s,a]$ by the powers of s. Notice that $([s,a]^s)^a = ([s,a])^a)^s$. So the subgroup K of H generated by the conjugates of the elements $h_1,\dots,h_m$ by the powers of a is again finitely generated. But $K=[H,H]$. Thus H is an extension of one finitely generated abelian group by another finitely generated abelian group and therefore polycyclic. □

We come now to the proof of Theorem 1.3. We will restrict our attention to the case of a two-generator metabelian group G since the general case follows along the same lines. To this end then notice that if G is generated by x_1,x_2 then $\gamma_n(G)/\gamma_{n+1}(G)$ is generated by the right-normed commutators of the form

$$[y_1,\dots,y_n]\gamma_{n+1},$$

where the $y_j\in\{x_1,x_2\}$ and $y_1=x_1, y_2=x_2$. Now $\widehat{G}$ is metabelian. So in order to prove that the finitely generated subgroups of $\widehat{G}$ are polycyclic, it suffices to prove that the two-generator subgroups of $\widehat{G}$ are polycyclic. Consequently it is, by Lemma 3.2, sufficient to prove that if $s,a\in\widehat{G}$ and $H=gp(s,a)$, then the subgroups of H generated by the conjugates of $b=[s,a]$ by the powers of both s and a is finitely generated. We will prove that the subgroup B of H generated by the conjugates of b by the powers of s is finitely generated.

Lemma 3.3. *Let $s\in\widehat{G}$ and let $b\in[\widehat{G},\widehat{G}]$. Then the subgroup B of $\widehat{G}$ generated by the conjugates of b by the powers of s is a finitely generated abelian group.*

Proof. Now

$$s(n)=s_1\dots s_n\gamma_{n+1}(G),\ b(n)=b_1\dots b_n\gamma_{n+1}(G),$$

where here $s_j\in\gamma_j(G)$, $b_j\in\gamma_j(G)$ for each j. If $s_1\in\gamma_2(G)$, then s and b commute and so there is nothing to prove. We consider then the case where $s_1\notin\gamma_2(G)$. Now we need to consider the elements b_n. To this end, let us denote by Y_n the set of commutators of the form $z(y_1,\dots,y_n)=[x_1,x_2,y_1,\dots,y_{n-2}]$ of weight $n>1$ where we adopt the convention that if $n=2$, $z=z(y_1,y_2,\dots,y_{n-2})=[x_1,x_2]$. Now we have already proved that there exist two polynomials α,β in s_1,s_1^{-1}, where

$$\alpha=c_0+c_1s_1+\dots+c_{m-1}{s_1}^{m-1}-s_1^m$$

and

$$\beta=-{s_1}^{-n}+d_{n-1}{s_1}^{-n+1}+\dots+d_1{s_1}^{-1}$$

such that

$$[x_2,x_1]\alpha=[x_2,x_1]\beta=0.$$

Now each $z(n)$ can be rewritten using exponential notation as

$$[x_1, x_2]^{(y_1-1)\dots(y_{n-2}-1)}.$$

So it follows that the action of α on z_n can be re-expressed as follows:

$$([x_2, x_1]^{(y_1-1)\dots(y_{n-2}-1))})^\alpha = [x_2, x_1]^{\alpha(y_1-1)\dots(y_{n-2}-1)}.$$

So

$$([x_2, x_1]^{(y_1-1)\dots(y_{n-2}-1))})^\alpha = 1.$$

It follows that $b\alpha = 0$ and similarly that $b\beta = 0$. Thus the conjugates of b by the powers of s generate a finitely generated group. This completes the proof. □

We can now complete the proof of Theorem 1.3.

The same proof used above can be used to prove that the conjugates of b by the powers of a is also generate a finitely generated group. So Lemma 3.2 applies as noted above. Thus we have proved that if G is polycyclic, the two-generator subgroups of $\widehat{G}$ are polycyclic and hence the finitely generated subgroups of $\widehat{G}$ are also polycyclic, as claimed. □

4. Detecting residual nilpotence

In order to explore para-equivalence of groups, we first give conditions for determining if a group is residually nilpotent. We use the notation $\gamma_\omega(G) := \cap_{i=1}^{\infty}\gamma_i(G)$.

Proposition 4.1. *Let G be a finitely generated group and let A be an abelian normal subgroup of G such that the quotient G/A is nilpotent. Then*

$$\gamma_\omega(G) = \{m \in A \mid m(1-g) = 0, \text{ for some } g \in G/A\},$$

where the action of G/A on A is defined via conjugation in G.

Proof. Let I be the augmentation ideal in $\mathbb{Z}[G/A]$. First observe that there exists an increasing function $k(n)$, such that, for every $n \geq 2$,

$$\gamma_{k(n)}(G) \subseteq AI^n \subseteq \gamma_{n+1}(G)$$

This follows from the following property: let Q be a group, such that $H \lhd Q$ is a nilpotent subgroup, Q/H is nilpotent and the corresponding action of Q/H on H is nilpotent, then Q is nilpotent (see, for example, [Hi], section 3). Hence

$$\gamma_\omega(G) = \bigcap_n AI^n.$$

Recall that, for a finitely generated nilpotent group H, the augmentation ideal I of the group ring $\mathbb{Z}[H]$ has the Artin-Rees property [Sm]. That is, for every finitely generated $\mathbb{Z}[H]$-module M and a submodule $N \subset M$, the I-adic topology on N coincides with the restriction on N of the I-adic topology on M. In particular, for every finitely generated $\mathbb{Z}[H]$-module M,

$$\bigcap_n MI^n = \{m \in M \mid m(1-a) = 0 \text{ for some } a \in I\}.$$

Applying this property to the case $M = A$ and $H = G/A$, we obtain the needed result. □

5. Examples

5.1. **Example.** Our first example is a family of residually nilpotent groups such that each group is the unique group in its para-equivalence class. In fact, more is true. The family of groups below are determined by their lower central *sequence.* To our knowledge, these are the first examples of non-nilpotent groups which are determined by their lower central sequence.

Theorem 5.1. *Suppose that $\Gamma_n = \langle a, b \mid a^b = a^n \rangle$, $n \neq 2$. Let H be a finitely generated, residually nilpotent group. Suppose further that for each k there is an isomorphism*

$$\Gamma_n/\gamma_k(\Gamma_n) \cong H/\gamma_k(H).$$

Then $H \cong \Gamma_n$.

The above groups are residually nilpotent by Proposition 4.1.

Remark 5.2. *We do not assume in the above theorem that H is para-Γ_n, nor even that there is an isomorphism of lower central quotient towers of groups. Only that for each k, there is an isomorphism of groups $\Gamma_n/\gamma_k(\Gamma_n) \cong H/\gamma_k(H)$.*

Question. *If two groups have the same lower central sequence, are their lower central series quotient towers isomorphic?*

Remark 5.3. *We can further weaken the condition of residual nilpotence in Theorem 5.1. In fact, if H is a finitely generated, transfinitely nilpotent group with the same lower central sequence as Γ_n $(n \neq 2)$, then $H \cong \Gamma_n$.*

Proof of Remark 5.3. Let H be a transfinitely nilpotent group with the same lower central sequence as Γ_n $(n \neq 0, 2)$. Then $H/\gamma_\omega(H)$ also has the same lower central sequence as Γ_n, and by Theorem 5.1, $H/\gamma_\omega(H) \cong \Gamma_n$.

Consider the following central extension of Γ_n:

$$1 \to \gamma_\omega(H)/[\gamma_\omega(H), H] \to H/[\gamma_\omega(H), H] \to \Gamma_n \to 1 \tag{5.1}$$

The following statement is proved in [Mi]: If G be a one-relator group and

$$1 \to N \to \tilde{G} \to G \to 1$$

is a central extension of G, then G is residually nilpotent if and only if $\tilde{G}$ is. This result applied to extension (5.1) implies that the group $H/[\gamma_\omega(H), H]$ is residually nilpotent, hence $\gamma_\omega(H) = [\gamma_\omega(H), H]$ and the transfinite nilpotence of H, together with Theorem 5.1, imply that $H \cong \Gamma_n$. □

Proof of Theorem 5.1. We can assume that $n \geq 2$, since the case $n < 0$ can be proved in the same way. Suppose that H is a residually nilpotent group with the same lower central sequence as Γ_n $(n \neq 2)$. For convenience, put $G = \Gamma_n$. Since H embeds in its pronilpotent completion, it follows that H is metabelian. Moreover $H/\gamma_2(H) \cong G/\gamma_2(G)$. So there exist elements t and s in H which generate H modulo $\gamma_2(H)$ with t of order $n-1$ modulo $\gamma_2(H)$ and s of infinite order modulo $\gamma_2(H)$. Now observe that if $c \in [G, G]$, then $b^{-1}cb = c^n$.

Now we will prove by induction on k the following statements:

(1) $s^{-1}ts = t^n$ modulo $\gamma_k(H)$;

(2) The quotient $H/\gamma_k(H)$ has presentation

$$\langle \bar{s}, \bar{t} \mid \bar{s}^{-1}\bar{t}\bar{s} = \bar{t}^n,\ \gamma_k\rangle$$

where $\bar{s}, \bar{t}$ are cosets $s \cdot \gamma_k(H)$ and $t \cdot \gamma_k(H)$ respectively.

Clearly, (2) implies (1), however, our proof will be constructed in another direction. The statements (1) and (2) hold for $k = 2$ by the definition of the elements s and t, and using the isomorphism $H/\gamma_2(H) \cong G/\gamma_2(G)$.

Now assume that (1) and (2) hold for a given k. The isomorphism

$$G/\gamma_{k+1}(G) \cong H/\gamma_{k+1}(H)$$

implies that there are elements $u, v \in H$, such that $H/\gamma_{k+1}(H)$ has a presentation

$$\langle \bar{u}, \bar{v} \mid \bar{u}^{-1}\bar{v}\bar{u} = \bar{v}^n,\ \gamma_{k+1}\rangle$$

where $\bar{u}, \bar{v}$ are cosets $u \cdot \gamma_{k+1}(H)$ and $v \cdot \gamma_{k+1}(H)$ respectively. The elements of the group $H/\gamma_{k+1}(H)$ have a normal form of the type:

$$g = \bar{u}^k \bar{v}^l,\ k, \ell \in \mathbb{Z}.$$

Present the cosets $s \cdot \gamma_{k+1}(H),\ t \cdot \gamma_{k+1}(H)$.

$$s \cdot \gamma_{k+1}(H) = \bar{u}^{k_1}\bar{v}^{\ell_1} \tag{5.2}$$

$$t \cdot \gamma_{k+1}(H) = \bar{u}^{k_2}\bar{v}^{\ell_2}. \tag{5.3}$$

Working modulo $\gamma_k(H)$, we get

$$s \equiv u^{k_1}v^{\ell_1} \mod \gamma_k(H)$$
$$t \equiv u^{k_2}v^{\ell_2} \mod \gamma_k(H).$$

Since t has order $(n-1)^{k-1}$ modulo $\gamma_k(H)$, $k_2 = 0$ and $(\ell_2, n-1) = 1$. Since u, v as well as s, t generate H modulo $\gamma_k(H)$, u can be presented in terms of s, t modulo $\gamma_k(H)$. Hence $k_1 = \pm 1$. Now observe that

$$s^{-1}ts \equiv (u^{k_1}v^{\ell_1})^{-1}v^{\ell_2}u^{k_1}v^{\ell_1} \equiv v^{\ell_2 u^{k_1}} \equiv t^n \equiv v^{\ell_2 n} \mod \gamma_k(H)$$

Case I: $k_1 = -1$. We have

$$v^{\ell_2 u^{-1}} \equiv v^{\ell_2 n} \mod \gamma_k(H)$$

and therefore

$$v^{\ell_2 n u} \equiv v^{\ell_2 n^2} \equiv v^{\ell_2} \mod \gamma_k(H)$$

We conclude that $v^{\ell_2(n^2-1)} \in \gamma_k(H)$. Since $(\ell_2, n-1) = 1$ and v has order $(n-1)^{k-1}$, for $k > 2$ this is possible only for $k \leq 3$ and $n = 3$. We consider the case $k = 3$, the case $k = 2$ is simpler. In this case, $v^8 \in \gamma_4(H), v^{u^{-1}} \equiv v^u \equiv v^3 \mod \gamma_4(H)$ and

$$s^{-1}ts \equiv v^{-\ell_1}uv^{\ell_2}u^{-1}v^{\ell_1} \equiv v^{\ell_2 n} \equiv t^n \mod \gamma_{k+1}(H)$$

Case II: $k_1 = 1$. We have

$$s^{-1}ts \equiv v^{-\ell_1}u^{-1}v^{\ell_2}uv^{\ell_1} \equiv v^{\ell_2 n} \equiv t^n \mod \gamma_{k+1}(H).$$

This finishes the proof of (1) for $k+1$. Now we will prove (2) for $k+1$. We return to the presentation (5.2)-(5.3). Since $k_2 = 0$ and $(\ell_2, n-1) = 1$, we conclude that u and v can be expressed in terms of s, t in H modulo $\gamma_{k+1}(H)$, i.e. the group $H/\gamma_{k+1}(H)$ is generated by cosets $s \cdot \gamma_{k+1}(H), t \cdot \gamma_{k+1}(H)$. Now consider the homomorphism

$$f : G/\gamma_{k+1}(G) \to H/\gamma_{k+1}(H)$$

given by setting

$$b \cdot \gamma_{k+1}(G) \mapsto s \cdot \gamma_{k+1}(H)$$
$$a \cdot \gamma_{k+1}(G) \mapsto t \cdot \gamma_{k+1}(H).$$

Since $s\cdot\gamma_{k+1}(H), t\cdot\gamma_{k+1}(H)$ generate $H/\gamma_{k+1}(H)$, the map f is an epimorphism. However, the group $G/\gamma_{k+1}(G)$ is Hopfian, i.e. every epimorphism $G/\gamma_{k+1}(G) \to G/\gamma_{k+1}(G)$ is an isomorphism. This shows that f is an isomorphism and (2) is proved for $k+1$. This completes the induction on k.

Now G is defined by the single relation $b^{-1}ab = a^n$ and so the map which sends a to b and t to s can be continued to a homomorphism ϕ of G into H. The homomorphism ϕ induces isomorphisms between the corresponding terms of the lower central sequences of G and H and so H is para-G. Since G is torsion-free metabelian, it follows that H is also torsion-free metabelian. Now choose, if possible, a finite set of generators for H which includes s, t and finitely many elements $c_1, \dots c_k$ with $k > 1$, which lie in the derived group of H. Notice that the elements $t, c_1, \dots, c_k$ commute and so we can assume that they freely generate a free abelian group. Moreover

$$s^{-1}ts = t^n, s^{-1}c_1 s = c_1^n, \dots, s^{-1}c_k s = c_k^n.$$

But this implies that $H/\gamma_2(H)$ contains the direct product of two cyclic groups of order $n-1$, which is not possible. So this completes the proof. □

5.2. **Example.** Recall from Theorem 2.11 that if $G \to H$ is a para-equivalence of finitely generated groups, then G is isomorphic to a subgroup of H of finite index, and H is isomorphic to a subgroup of G of finite index.

This does not imply $G \cong H$ as the next example, and Proposition 5.4 shows.

Consider the group ring of the group $\mathbb{Z}$, $\mathbb{Z}[\mathbb{Z}]$. By writing $\mathbb{Z}$ multiplicatively as $\mathbb{Z} = \{t^n \mid n \in \mathbb{Z}\}$, we have an isomorphism between the group ring $\mathbb{Z}[\mathbb{Z}]$ and the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$.

Let A be the non-principal ideal $A = (2t-1, 2-t) \subset \mathbb{Z}[t, t^{-1}]$, were this latter represents the Laurent polynomial ring on a single variable. Consider the groupa

$$G = \mathbb{Z}[t, t^{-1}] \rtimes \mathbb{Z} = \mathbb{Z} \wr \mathbb{Z} \qquad \text{and} \qquad H = A \rtimes \mathbb{Z}$$

and the natural map

$$f : H \to G,$$

induced by inclusion of $\mathbb{Z}[t, t^{-1}]$-modules $A \hookrightarrow \mathbb{Z}[t, t^{-1}]$.

$G \neq H$, since $[G, G] = AI \ncong \mathbb{Z}[t, t^{-1}]I = [H, H]$, since $A \cdot (t-1)$ is not a principal ideal.

Note that G is residually nilpotent by Lemma 2.4 since $\mathbb{Z}[t, t^{-1}]$ is an integral domain, and clearly torsion-free.

Proposition 5.4. *H and G are non-isomorphic para-equivalent groups. More precisely, the homomorphism $f : H \to G$ induces isomorphisms of lower central quotients.*

Proof. This is immediate from Lemma 2.4 since the composition of any two successive homomorphisms below

$$\mathbb{Z}[t, t^{-1}] \xrightarrow{\beta} (2t-1, 2-t) \subset \mathbb{Z}[t, t^{-1}] \xrightarrow{\beta} (2t-1, 2-t)$$

is given by multiplication by $2t - 1 \in S$, where β is the $Z[t, t^{-1}]$-module homomorphism given by $\beta(1) = 2t - 1$. This implies these modules have the same I-adic quotients,

which implies the $H \to G$ induces an isomorphism on lower central series quotients. In particular, by Theorem 2.11, $H \to G$ is a para-equivalence.

$\square$

5.3. **Example.** Even if G and H are polycyclic groups, para-equivalence does not imply the groups are isomorphic. The following modification of the above tools can be used, along with ideal class theory, to exhibit many examples. We offer one example here. A forthcoming paper will more thoroughly examine the connection to class field theory, as well as give a classification theorem for isomorphism classes of para-equivalent groups [BMO2].

Theorem 5.5. *There is a residually nilpotent, polycyclic, metabelian group G such that $[G, G]$ is finitely generated abelian, and a non-isomorphic, metabelian, polycyclic para-equivalent group H.*

Remark 5.6. *Ideal class theory inspired this example. It's well known that the ideal class group of $\mathbf{Q}(\zeta_{23})$ has order 3 and is generated by the non-principal ideal $(2, 1+P) \subset \mathbb{Z}[\zeta_{23}]$, where P is the Guassian period described in the proof below. (See, for instance,* [Ma, page 86]*.)*

Proof. Consider the Dedekind domain

$$\mathbb{Z}[\zeta_{23}] \cong \mathbb{Z}[t, t^{-1}]/(N(t)) \text{ where } N(t) = \Sigma_{k=0}^{22} t^k.$$

We view $\mathbb{Z}[\zeta_{23}]$ as a $\mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$-module via the isomorphism above. Let

$$G = \mathbb{Z}[\zeta_{23}] \rtimes \mathbb{Z}$$

where the quotient group acts on $\mathbb{Z}[\zeta_{23}]$ by multiplication by t. Note that augmentation ϵ for $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$ determines a commutative diagram where C_{23} denotes the cyclic group with 23 elements.

$$\begin{tikzcd}
\mathbb{Z}[t, t^{-1}] \arrow[r, "\epsilon"] \arrow[d, "q"] & \mathbb{Z} \arrow[d] \\
\mathbb{Z}[\zeta_{23}] \arrow[r] & C_{23}
\end{tikzcd}$$

Therefore if $p(t) \in \mathbb{Z}[t, t^{-1}]$ and $p(1) = \pm 1$, then $q(p(t)) \neq 0$. Hence, since $\mathbb{Z}[\zeta_{23}]$ is an integral domain, the multiplicative set $S \subset \mathbb{Z}[\zeta_{23}]$, the image of $1 + \ker\{\epsilon\colon \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}\}$ in $\mathbb{Z}[\zeta_{23}]$, is invertible. By Theorem 4.1, G is residually nilpotent, as well as finitely generated and metabelian.

We now construct a residually nilpotent, metabelian group, H, and a para-equivalence $G \to H$ such that G and H are not isomorphic.

Consider the *Gaussian period*

$$P = \Sigma_{k=1}^{11} \zeta_{23}^{k^2} = \zeta_{23} + \zeta_{23}^2 + \zeta_{23}^3 + \zeta_{23}^4 + \zeta_{23}^6 + \zeta_{23}^8 + \zeta_{23}^9 + \zeta_{23}^{12} + \zeta_{23}^{13} + \zeta_{23}^{16} + \zeta_{23}^{18} = \frac{-1 + \sqrt{-23}}{2},$$

and the element

$$1 + P = \frac{1 + \sqrt{-23}}{2} \subset \mathbb{Z}[\zeta_{23}] \subset \mathbb{C}.$$

Now consider the non-principal ideal

$$(2, 1+P) \subset \mathbb{Z}[\zeta_{23}].$$

Let H be the group

$$H = (2, 1+P) \rtimes \mathbb{Z}$$

and observe that the inclusion $(2, 1+P) \subset \mathbb{Z}[\zeta_{23}]$ induces an inclusion of groups $H \subset G$.

We now construct an element $s \in S$ such that the principle ideal $(s) \subset \mathbb{Z}[\zeta_{23}]$ satifies

$$(s) \subset (2, 1+P).$$

Let

$$p(t) = 2(1 + t + t^2 + t^3 + t^4 + t^6 + t^8 + t^9 + t^{12} + t^{13} + t^{16} + t^{18}) - N(t).$$

Then

$$p(t) \in 1 + \ker\{\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}\}$$

since $p(1) = 1$. Let $s = p(\zeta_{23}) \in S$. Also,

$$p(\zeta_{23}) = 2 + 2P = 1 + \sqrt{-23} \in \mathbb{Z}[\zeta_{23}].$$

Now consider the following diagram where α is the homomorphism such that $\alpha(1) = s$, that is $\alpha(1) = 2(1+P)$.

$$\mathbb{Z}[\zeta_{23}] \xrightarrow{\alpha} (2, 1+P) \xrightarrow{\subset} \mathbb{Z}[\zeta_{23}] \xrightarrow{\alpha} (2, 1+P)$$

Each composition of two homomorphisms is given by multiplication by $s \in S$. Hence, all homomorphisms in this diagram induce isomorphisms on I-adic quotients by Lemma 2.4.

The homomorphisms in the corresponding diagram of groups induces isomorphisms on lower central series quotients.

$$G \xrightarrow{\alpha \rtimes id} H \subset G \xrightarrow{\alpha \rtimes id} H$$

and

$$G \xrightarrow{\alpha \rtimes id} H$$

is a para-equivalence.

However

$$G = \mathbb{Z}[\zeta_{23}] \rtimes \mathbb{Z} \ncong (2, 1+P) \rtimes \mathbb{Z} = H$$

since $(2, 1+P)$ is not a principal ideal in $\mathbb{Z}[\zeta_{23}]$, and therefor not isomorphic to $\mathbb{Z}[\zeta_{23}]$. $\square$

5.4. **Example.** Recall that finitely generated, metabelian, para-G groups are para-equivalent by Theorem 2.11. Observe that, for infinitely generated metabelian groups, para-G does not imply para-equivalence, since $\widehat{G}$ is a para-G group. The following shows that para-equivalence equals para-G can fail when G is metabelian, and even for G with $[G, G]$ a free abelian group.

Proposition 5.7. *Let C be an infinite cyclic group, $G = C \wr C$ and H be a residually nilpotent para-G group. If H is countable then $[H, H]$ is free abelian. There exists an uncountable example of H with non-free abelian $[H, H]$.*

Proof. Consider the free nilpotent completion $\widehat{G}$ of G. For every $n \geq 2$, we have the following diagram

$$\begin{tikzcd}
1 \arrow[r] & C^n \arrow[r] \arrow[d] & G/\gamma_{n+1}(G) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & C^{n-1} \arrow[r] & G/\gamma_n(G) \arrow[r] & C \arrow[r] & 1
\end{tikzcd}$$

where the left hand vertical arrow is projection onto the first $n-1$ coordinates. Since the spectrum of epimorphisms satisfy the Mittag-Leffler condition, one can take the inverse limit by n and obtain the following short exact sequence:

$$1 \to C^{\mathbb{N}} \to \widehat{G} \to C \to 1$$

Here $C^{\mathbb{N}}$ is the Baer-Specker group (see [Bae], [Sp]), i.e. the group of all integer sequences with component-wise addition.

Now let H be a countable residually nilpotent para-G-group. Since G is metabelian, $[H,H]$ is an abelian group and there is a monomorphism

$$[H,H] \hookrightarrow [\widehat{H},\widehat{H}] \simeq [\widehat{G},\widehat{G}] \subset C^{\mathbb{N}},$$

where $\widehat{H}$ is the free nilpotent completion of H. Every countable subgroup of $C^{\mathbb{N}}$ is a free abelian [Sp], hence $[H,H]$ is a free abelian. As an example of an uncountable para-G-group with non-free commutator subgroup one can take $\widehat{G}$ itself, its commutator subgroup is not free, since $C^{\mathbb{N}}$ is not a free abelian group [Bae]. $\square$

References

[AM] M. F. Atiyah and I. G. MacDonald: Introduction to commutative algebra, *Addison Wesley Publishing Co., Reading Mass.-London.Don Mills, Ont.* (1969)

[Bae] R. Baer: Abelian groups without elements of finite order, *Duke Math. J.* **3** (1937), 68–122.

[Bau1] G. Baumslag: Groups with the same lower central sequence as a relatively free group. I. The groups. *Trans. Amer. Math. Soc.* **129** (1967), 308–321.

[Bau2] G. Baumslag: Groups with the same lower central sequence as a relatively free group. II. Properties. *Trans. Amer. Math. Soc.* **142** (1969), 507–538.

[Bau3] G. Baumslag: On the Residual Nilpotence of Certain One-Relator Groups, *Comm. Pure Appl. Math.* **21** (1968), 491–506.

[BMO1] G. Baumslag, R. Mikhailov and K. Orr: A new look at finitely generated metabelian groups, Contemporary Mathematics: Combinatorial and Computational Group Theory with Cryptography, **582**, (2012).

[BMO2] G. Baumslag, R. Mikhailov and K. Orr: Ideal class theory and metabelian groups.

[BSta] G. Baumslag and U. Stammbach: On the inverse limit of free nilpotent groups, *Comm. Math. Helv.* **52** (1977), 219–233.

[BStr] R. Bieri and R. Strebel: A geometric invariant for modules over an abelian group. *J. Reine Angew. Math.* **322** (1981), 170–189.

[Bo1] A.K. Bousfield: Homological localization towers for groups and Π-modules. *Mem. Amer. Math. Soc.* **10**, (1977).

[BR] M. Bridson and A. Reid: Nilpotent completions of groups, Grothendieck pairs and four problems of Baumslag, arxiv: 1211.0493.

[CO] T. Cochran and K. Orr: Kent E. Stability of lower central series of compact 3-manifold groups, *Topology* **37** (1998), 497–526.

[E] D. Eisenbud: Commutative Algebra. *Graduate Texts in Mathematics*, **150**, (1995).

[Ha] P. Hall: Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* **2** (1958), 787–801.

[Hi] P. Hilton: Nilpotent actions on nilpotent groups, *Lecture Notes in Mathematics*, **450** (1975), 174–196.

[KM] O. Kharlampovich, and A. Myasnikov: Elementary theory of free non-abelian groups, *J. Algebra*, **302** (2006), 451BЦ-552.

[L1] J. P. Levine: Link concordance and algebraic closure of groups, *Comment. Math. Helv.* 64 (1989), no. 2, 236–255.

[L2] J. P. Levine: Link concordance and algebraic closure. II, *Invent. Math.* 96 (1989), no. 3, 571–592.

[L3] J. P. Levine: Link invariants via the eta invariant, *Comment. Math. Helv.* 69 (1994), no. 1, 82–119.

[Ma] D. Marcus: Number fields, Universitext. Springer-Verlag, New York-Heidelberg, (1977.)

[Mi] R. Mikhailov: Residual nilpotence and residual solvability of groups, *Sb. Math.* **196** (2005), 1659–1675.

[Se] D. Segal: On Abelian-by-polycyclic groups, *J.London Math. Soc.* **11**, (1975), 445–452.

[Sela] Z. Sela: Diophantine geometry over groups. I. Makanin-Razborov diagrams, *Publ. Math. IHES* **93**, 31–105.

[Sm] P.F. Smith: The Artin-Rees property, *Lecture Notes in Math.*, **924**, Springer, Berlin-New York, (1982), 197–240.

[Sp] E. Specker: Additive Gruppen von Folgen ganzer Zahlen, *Port. Math.* **9** (1949), 131–140.

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